Three-Component Generalization of Burgers Equation and its Bi-Hamiltonian Structures

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Abstract: A hierarchy of three-component generalization of Burgers equation, which is associated with a 3×3 matrix eigenvalue problem, is generated by using the zero-curvature equation. By means of the trace identity, the bi-Hamiltonian structures of this hierarchy are constructed. Moreover, the infinite conservation laws for the hierarchy are obtained with the aid of spectral parameter expansion.

Key words: Three-component generalization of Burgers equation; Bi-Hamiltonian structures; Conservation laws.

1 Introduction

It is well known that soliton equations, a kind of nonlinear evolution equations, are infinite dimensional integrable systems. Searching for new integrable models has been an hot topic in mathematical physics, lots of famous soliton equations are found (see e.g. Refs. [1-5] and the references therein). Among the numerous equations, Burgers equation has

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attracted significant attention. The Burgers equation was first proposed by Burgers [6,7] as a model in the study of turbulence. It can be widely applied in various fields of science, such as fluid mechanics, aerodynamics, heat conduction and traffic flow, etc [8-11]. In the present paper, our study is focus on a hierarchy of new nonlinear evolution equations which can be reduced to the famous Burgers equation. The Hamiltonian structures and conservation laws of this hierarchy will be studied depending on the trace identity [12-14] and its Lax pairs [15-19].

In this paper, we propose a hierarchy of new nonlinear evolution equations which is related to a 3×3 matrix spectral problem with three potentials. The first nontrivial member in this hierarchy is a three-component generalization of Burgers equation

$$\begin{aligned} u_t &= (3w_x - uw + 2v)_x, \\ v_t &= (2w_{xx} + v_x)_x - 2uw_{xx} - 2vw_x - v_x w - uv_x, \\ w_t &= (-w_x - 2w^2 - uw)_x, \end{aligned} \tag{1}$$

as $w = 0$, Eq. (1) can be reduced to the Burgers equation [6,7]

$$u_t = u_{xx} - uu_x. \tag{2}$$

The arrangement of this paper is as follows. In the following section, we obtain a hierarchy of three-component generalization of Burgers equation with the help of the zero-curvature equation and the Lenard recursion equations. Then in Sec. 3, the generalized bi-Hamiltonian structures are established based on trace identity. Sec. 4 is devoted to giving the infinite conservation laws of the first two nontrivial equation.

2 Three-Component Generalization of Burgers Equation

In this section, we shall derive a hierarchy of three-component generalization of Burgers equation associated with the following 3×3 matrix spectral problem

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + v & u & \lambda \\ w & 0 & 0 \end{pmatrix}, \tag{3}$$

where u, v, and w are three potentials, and λ is a constant spectral parameter. First, we consider the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = (V_{ij})_{3\times 3}, \tag{4}$$

which is equivalent to

$$\begin{aligned}
&V_{11,x} - V_{21} + (\lambda + v)V_{12} + wV_{13} = 0,\\
&V_{12,x} + V_{11} + uV_{12} - V_{22} = 0,\\
&V_{13,x} + \lambda V_{12} - V_{23} = 0,\\
&V_{21,x} - uV_{21} - \lambda V_{31} + (\lambda + v)(V_{22} - V_{11}) + wV_{23} = 0,\\
&V_{22,x} - (\lambda + v)V_{12} - \lambda V_{32} + V_{21} = 0,\\
&V_{23,x} - (\lambda + v)V_{13} - uV_{23} + \lambda(V_{22} - V_{33}) = 0,\\
&V_{31,x} + (\lambda + v)V_{32} + w(V_{33} - V_{11}) = 0,\\
&V_{32,x} - wV_{12} + V_{31} + uV_{32} = 0,\\
&V_{33,x} - wV_{13} + \lambda V_{32} = 0.
\end{aligned} \tag{5}$$

If we define each entry $V_{ij}(B, C, D)$ as follows:

$$\begin{aligned}
&V_{11} = -2\partial B + (-\partial^2 + u\partial + v)C + \lambda(C + D), \quad V_{12} = B, \quad V_{13} = \lambda C,\\
&V_{21} = (-2\partial^2 + v)B + (\partial u\partial + \partial v - \partial^3)C + \lambda(B + wC + \partial C + \partial D),\\
&V_{22} = (-\partial + u)B + (-\partial^2 + u\partial + v)C + \lambda(C + D), \quad V_{23} = \lambda(B + \partial C),\\
&V_{31} = wB - (\partial w + uw)C + (\partial^2 + u\partial)D, \quad V_{32} = wC - \partial D, \quad V_{33} = \lambda D,
\end{aligned} \tag{6}$$

where $\partial = \partial/\partial x$, substituting (6) into (5) yields the Lenard equations

$$\begin{aligned}&(-3\partial^2 + \partial u)B + (-2\partial^3 + 2\partial u\partial + 2\partial v)C + \lambda(2\partial C + 3\partial D) = 0,\\&(-2\partial^3 + 2u\partial^2 + \partial v + v\partial)B + (-\partial^4 + u\partial^3 + \partial^2 u\partial + \partial^2 v - u\partial u\partial - u\partial v)C\\&+\lambda[2\partial B + (\partial^2 + 2\partial w + w\partial - u\partial)C - 2u\partial D] = 0,\\&(\partial w + 2w\partial)B + [w\partial^2 - \partial^2 w - \partial(uw) - uw\partial]C + (\partial^3 + \partial u\partial - v\partial)D - \lambda\partial D = 0.\end{aligned} \tag{7}$$

The functions B, C and D can be expanded into a Laurent series in λ:

$$B = \sum_{j\geq 0} B_j\lambda^{-j}, \quad C = \sum_{j\geq 0} C_j\lambda^{-j}, \quad D = \sum_{j\geq 0} D_j\lambda^{-j}. \tag{8}$$

Substituting (8) into (7), we obtain the Lenard equations

$$KG_j = JG_{j+1} \ \ (j \geq 0), \qquad JG_0 = 0, \tag{9}$$

where $G_j = (B_j, C_j, D_j)^{\mathrm{T}}$, K and J are two operators defined by

$$K = \begin{pmatrix} -3\partial^2 + \partial u & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ \partial w + 2w\partial & K_{32} & K_{33} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -2\partial & -3\partial \\ -2\partial & J_{22} & 2u\partial \\ 0 & 0 & \partial \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}K_{12} &= -2\partial^3 + 2\partial u\partial + 2\partial v,\\K_{21} &= -2\partial^3 + 2u\partial^2 + \partial v + v\partial,\\K_{22} &= -\partial^4 + u\partial^3 + \partial^2 u\partial + \partial^2 v - u\partial u\partial - u\partial v,\\K_{32} &= w\partial^2 - \partial^2 w - \partial(uw) - uw\partial,\\K_{33} &= \partial^3 + \partial u\partial - v\partial,\\J_{22} &= -\partial^2 + u\partial - 2\partial w - w\partial.\end{aligned}$$

It is easy to see that

$$\ker J = \{c_0 g_0 + \widehat{c}_0\widehat{g}_0 + \widetilde{c}_0\widetilde{g}_0 \quad | \quad \forall c_0, \widehat{c}_0, \widetilde{c}_0 \in \mathbb{R}\}$$

with

$$g_0 = (-w, 1, 0)^{\mathrm{T}}, \quad \widehat{g}_0 = (1, 0, 0)^{\mathrm{T}}, \quad \widetilde{g}_0 = (0, 0, 1)^{\mathrm{T}}, \tag{11}$$

where $\tilde{g}_0 \in \ker K$. To find a general representation of solutions of (9), we introduce two special Lenard recursion equations

$$Kg_j = Jg_{j+1}, \qquad K\widehat{g}_j = J\widehat{g}_{j+1}, \quad j \geq 0, \tag{12}$$

then we have

$$g_1 = \begin{pmatrix} -w_{xx} + 2vw - (uw)_x - 3ww_x - 3uw^2 - 4w^3 \\ 2uw - v + 3w^2 \\ -w_x - uw - 2w^2 \end{pmatrix}, \tag{13}$$

$$\widehat{g}_1 = \begin{pmatrix} \frac{3}{4}uw - \frac{1}{2}v + \frac{3}{4}w_x + \frac{1}{4}u_x - \frac{1}{8}u^2 + \frac{15}{8}w^2 \\ -\frac{1}{2}u - \frac{3}{2}w \\ w \end{pmatrix}. \tag{14}$$

It can be verified that $JG_0 = 0$ has a solution

$$G_0 = \alpha_0 g_0 + \widehat{\alpha}_0 \widehat{g}_0, \tag{15}$$

then we obtain a special solution

$$G_j = \alpha_0 g_j + \widehat{\alpha}_0 \widehat{g}_j, \tag{16}$$

where α_0 and $\widehat{\alpha}_0$ are arbitrary constants.

Now, we consider the auxiliary spectral problem

$$\psi_{t_m} = V^{(m)}\psi, \quad V^{(m)} = \Big(V_{ij}\big(B^{(m)}, C^{(m)}, D^{(m)}\big)\Big)_{3\times 3}, \tag{17}$$

where

$$B^{(m)} = \sum_{j=0}^{m} B_j \lambda^{m-j}, \quad C^{(m)} = \sum_{j=0}^{m} C_j \lambda^{m-j}, \quad D^{(m)} = \sum_{j=0}^{m} D_j \lambda^{m-j}. \tag{18}$$

The compatible condition of (3) and (17) yields the zero-curvature equation $U_{t_m} - V_x^{(m)} + \big[U, V^{(m)}\big] = 0$, which is equivalent to a hierarchy of nonlinear evolution equations

$$(u_{t_m}, v_{t_m}, w_{t_m})^{\mathrm{T}} = X_m = KG_m = JG_{m+1}, \qquad m \geq 0. \tag{19}$$

Especially, for $\alpha_0 = 1, \widehat{\alpha}_0 = 0, m = 0$, we have the three-component generalization of Burgers equation

$$
\begin{aligned}
u_{t_0} &= 3w_{xx} - u_x w - u w_x + 2v_x, \\
v_{t_0} &= 2w_{xxx} + v_{xx} - 2u w_{xx} - 2v w_x - v_x w - u v_x, \\
w_{t_0} &= -w_{xx} - 4w w_x - u_x w - u w_x,
\end{aligned} \tag{20}
$$

which is equivalent to Eq. (1) for $t_0 = t$. It is easy to see that Eq. (1) has the Lax pair

$$
\begin{cases} \psi_x = U\psi, \\ \psi_t = V\psi, \end{cases} \qquad \psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}
$$

where

$$
U = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + v & u & \lambda \\ w & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 2w_x + v + \lambda & -w & \lambda \\ 2w_{xx} - vw + v_x & w_x - uw + v + \lambda & -\lambda w \\ -w_x - uw - w^2 & w & 0 \end{pmatrix}.
$$

For $w = 0$, Eq. (20) is reduced to Burgers equation (2).

The second member as $\alpha_0 = 0, \widehat{\alpha}_0 = 1$ in the hierarchy (19) is

$$
\begin{aligned}
u_{t_1} = {} & \left(\frac{3}{4}w_{xx} + \frac{1}{4}u_{xx} - \frac{9}{4}u_x w - \frac{9}{2}u w_x + \frac{3}{2}v_x - \frac{45}{4}w w_x - \frac{3}{2}uv - 3vw - \frac{1}{8}u^3 + \frac{15}{8}uw^2 + \frac{3}{4}u^2 w\right)_x, \\
v_{t_1} = {} & \left(v_{xx} - \frac{3}{2}u_{xx}w - 3u w_{xx} - \frac{15}{2}w w_{xx} - \frac{9}{2}u_x w_x - \frac{1}{2}u v_x - \frac{3}{2}v_x w - \frac{3}{4}v w_x - \frac{1}{4}u_x v - \frac{1}{8}u^2 v \right. \\
& \left. - \frac{15}{2}w_x^2 - \frac{3}{4}v^2\right)_x + \frac{3}{2}u\left(u_{xx}w + 2u w_{xx} + 5w w_{xx} + 5w_x^2 + 3u_x w_x\right) + \frac{3}{4}v w_{xx} + \frac{1}{4}u_{xx}v - u v_{xx} \\
& + \frac{9}{4}u v_x w + \frac{3}{2}u_x v w + 3u v w_x + \frac{1}{4}u u_x v + \frac{15}{8}v_x w^2 + \frac{15}{2}v w w_x + \frac{1}{2}u^2 v_x, \\
w_{t_1} = {} & w_{xxx} + \frac{3}{2}u w_{xx} + \frac{15}{4}w w_{xx} + \frac{3}{4}u_{xx}w + \frac{15}{2}u w w_x - \frac{3}{2}v_x w + \frac{3}{4}u u_x w - \frac{3}{2}v w_x + \frac{9}{4}u_x w_x \\
& + \frac{3}{8}u^2 w_x + \frac{15}{4}u_x w^2 + \frac{105}{8}w^2 w_x + \frac{15}{4}w_x^2.
\end{aligned} \tag{21}
$$

3 The Generalized Bi-Hamiltonian Structures

In this section, our aim is to obtain the Hamiltonian forms of the hierarchy (19). By direct computing, we get

$$\begin{aligned}
&\operatorname{tr}\left(V\frac{\partial U}{\partial \lambda}\right) = V_{12} + V_{32} = B + wC - D_x, \\
&\operatorname{tr}\left(V\frac{\partial U}{\partial u}\right) = V_{22} = (uB - B_x) + (uC_x + vC - C_{xx}) + \lambda(C + D), \qquad (22)\\
&\operatorname{tr}\left(V\frac{\partial U}{\partial v}\right) = V_{12} = B, \quad \operatorname{tr}\left(V\frac{\partial U}{\partial w}\right) = V_{13} = \lambda C.
\end{aligned}$$

Using the trace identity [12-14], we have

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}\right)(B + wC - D_x)\big|_{\widehat{\alpha}_0=0} = \left[\lambda^{-\gamma_1}\frac{\partial}{\partial \lambda}\lambda^{\gamma_1}\right]\big(A, B, \lambda C\big)\big|_{\widehat{\alpha}_0=0} \qquad (23)$$

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}\right)(B + wC - D_x)\big|_{\alpha_0=0} = \left[\lambda^{-\gamma_2}\frac{\partial}{\partial \lambda}\lambda^{\gamma_2}\right]\big(A, B, \lambda C\big)\big|_{\alpha_0=0} \qquad (24)$$

where $A = uB - B_x + uC_x + vC - C_{xx} + \lambda(C + D)$, γ_1 and γ_2 are two constants to be fixed. Substituting (18) into (23) and (24), and equating the coefficients of λ^{-j}, we deduce that $\gamma_1 = -1, \gamma_2 = -\dfrac{1}{2}$ and

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}\right)(B_j + wC_j - D_{j,x})\big|_{\widehat{\alpha}_0=0} = (1 - j + \gamma_1)\big(A_j, B_{j-1}, C_j\big)\big|_{\widehat{\alpha}_0=0} \qquad (25)$$

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}\right)(B_j + wC_j - D_{j,x})\big|_{\alpha_0=0} = (1 - j + \gamma_2)\big(A_j, B_{j-1}, C_j\big)\big|_{\alpha_0=0} \qquad (26)$$

where

$$A_j = uB_{j-1} - B_{j-1,x} + uC_{j-1,x} + vC_{j-1} - C_{j-1,xx} + C_j + D_j.$$

From (25) and (26), we arrive at

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}, \frac{\delta}{\delta w}\right)^{\mathrm{T}} H_j = (A_j, B_{j-1}, C_j)^{\mathrm{T}}, \qquad j \geq 1. \qquad (27)$$

with

$$H_j = -\frac{1}{j}(B_j + wC_j - D_{j,x})\Big|_{\widehat{\alpha}_0=0} + \frac{2}{1-2j}(B_j + wC_j - D_{j,x})\Big|_{\alpha_0=0},$$

from which we obtain the bi-Hamiltonian [20,21] form of (19)

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_m} = \widetilde{K} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_m = \widetilde{J} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_{m+1}, \qquad m \geq 0, \tag{28}$$

where

$$\widetilde{K} = \begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} & \widetilde{K}_{13} \\ \widetilde{K}_{21} & \widetilde{K}_{22} & \widetilde{K}_{23} \\ \widetilde{K}_{31} & \widetilde{K}_{32} & \widetilde{K}_{33} \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} 6\partial & -3\partial^2 - 3\partial u & -2\partial \\ 3\partial^2 - 3u\partial & \widetilde{J}_{22} & u\partial - \partial^2 \\ -2\partial & \partial u + \partial^2 & 0 \end{pmatrix}, \tag{29}$$

$$\begin{aligned}
\widetilde{K}_{11} &= \frac{9}{2}\partial^3 - \frac{1}{2}\partial u \partial^{-1} u \partial, \\
\widetilde{K}_{12} &= -3\partial^4 - 3\partial^3 u + \partial u \partial^2 + \frac{3}{2}\partial^2 v + \frac{3}{2}\partial v \partial + \partial u \partial u - \frac{1}{2}\partial(uv) - \frac{1}{2}\partial u \partial^{-1} v \partial, \\
\widetilde{K}_{13} &= -2\partial^3 + 3\partial^2 w + \frac{3}{2}\partial w \partial + 2\partial u \partial - \partial(uw) + 2\partial v - \frac{1}{2}\partial u \partial^{-1} w \partial, \\
\widetilde{K}_{21} &= 3\partial^4 - 3u\partial^3 + \partial^2 u \partial - \frac{3}{2} v \partial^2 - \frac{3}{2}\partial v \partial - u \partial u \partial - \frac{1}{2} uv\partial - \frac{1}{2}\partial v \partial^{-1} u \partial, \\
\widetilde{K}_{22} &= -2\partial^5 - 2\partial^4 u + 2u\partial^4 + \partial^3 v + v\partial^3 + \partial^2 v \partial + \partial v \partial^2 + 2u\partial^3 u - u\partial^2 v + v\partial^2 u \\
&\quad - u\partial v \partial + \partial v \partial u - \frac{1}{2}\partial v^2 - \frac{1}{2} v^2 \partial - \frac{1}{2} v \partial v - \frac{1}{2}\partial v \partial^{-1} v \partial, \\
\widetilde{K}_{23} &= -\partial^4 + 2\partial^3 w + u\partial^3 + \partial^2 u \partial + \partial^2 w \partial - 2u\partial^2 w + \partial^2 v - u\partial u \partial - u \partial w \partial \\
&\quad - \partial(vw) - v\partial w - \frac{1}{2} vw\partial - u\partial v - \frac{1}{2}\partial v \partial^{-1} w \partial, \\
\widetilde{K}_{31} &= -2\partial^3 - 3w\partial^2 - \frac{3}{2}\partial w \partial - 2\partial u \partial - uw\partial + 2v\partial - \frac{1}{2}\partial w \partial^{-1} u \partial, \\
\widetilde{K}_{32} &= \partial^4 + 2w\partial^3 + \partial^3 u + \partial u \partial^2 + \partial w \partial^2 + 2w\partial^2 u - v\partial^2 + \partial u \partial u + \partial w \partial u - vw\partial \\
&\quad - w\partial v - \frac{1}{2}\partial(vw) - v\partial u - \frac{1}{2}\partial w \partial^{-1} v \partial, \\
\widetilde{K}_{33} &= w\partial^2 - \partial^2 w - \partial w^2 - w^2\partial - 2w\partial w - uw\partial - \partial(uw) - \frac{1}{2}\partial w \partial^{-1} w \partial, \\
\widetilde{J}_{22} &= -2\partial^3 - 2\partial^2 u + 2u\partial^2 + v\partial + \partial v + 2u\partial u.
\end{aligned}$$

In particular, Eq. (20) can be written as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_0} = \widetilde{K} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_0 = \widetilde{J} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_1, \tag{30}$$

with the Hamiltonian

$$H_0 = u + w, \quad H_1 = uw^2 + w^3 - vw - ww_x.$$

Eq. (21) can be written as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \widetilde{K} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_1 = \widetilde{J} \begin{pmatrix} \frac{\delta}{\delta u} \\ \frac{\delta}{\delta v} \\ \frac{\delta}{\delta w} \end{pmatrix} H_2, \tag{31}$$

with the Hamiltonian

$$\begin{aligned} H_1 &= v - \frac{1}{2}uw - \frac{1}{2}u_x + \frac{1}{2}w_x + \frac{1}{4}u^2 - \frac{3}{4}w^2, \\ H_2 &= -\frac{2}{3}\big(B_2 + wC_2 - D_{2,x}\big). \end{aligned}$$

4 Conservation Laws

In order to obtain the infinite conservation laws which correspond to the hierarchy (19), we set

$$\phi_1 = \frac{\psi_1}{\psi_3}, \quad \phi_2 = \frac{\psi_2}{\psi_3}, \tag{32}$$

then the spectral problem (3) is reduced to Riccati type equations

$$\phi_{1,x} + w\phi_1^2 - \phi_2 = 0, \quad \phi_{2,x} - v\phi_1 - u\phi_2 - \lambda\phi_1 - \lambda + w\phi_1\phi_2 = 0. \tag{33}$$

Expanding ϕ_1 and ϕ_2 as series in powers of λ

$$\phi_1 = \sum_{j=0}^{\infty} \kappa_j \lambda^{-j}, \quad \phi_2 = \sum_{j=0}^{\infty} \chi_j \lambda^{-j}, \tag{34}$$

then substituting (34) into (33) and comparing the coefficients of λ, we obtain

$$\begin{aligned} &\kappa_0 = -1, \quad \kappa_1 = w_x + v - uw - w^2, \\ &\kappa_2 = w_{xxx} + v_{xx} - (uw)_{xx} - 3(wv)_x + 3u_xw^2 + 9uww_x + 11w^2w_x - 4w_x^2 - 5ww_{xx} \\ &\qquad - v^2 + 3uvw + 4w^2v - uw_{xx} - uv_x + uu_xw + u^2w_x - 2u^2w^2 - 5uw^3 - 3w^4, \\ &\chi_0 = w, \quad \chi_1 = w_{xx} + v_x - (uw)_x - 4ww_x - 2wv + 2uw^2 + 2w^3, \end{aligned} \tag{35}$$

the recursion formulas for κ_j and χ_j $(j \geq 0)$

$$\begin{aligned}
\chi_j &= \kappa_{j,x} + w \sum_{i=0}^{j} \kappa_i \kappa_{j-i}, \\
\kappa_{j+1} &= \chi_{j,x} - v\kappa_j - u\chi_j + w \sum_{i=0}^{j} \kappa_j \chi_{j-i}.
\end{aligned} \tag{36}$$

On the other hand, it is easy to see that

$$\frac{\partial}{\partial t_m} \frac{\psi_{3,x}}{\psi_3} = \frac{\partial}{\partial x} \frac{\psi_{3,t}}{\psi_3}, \tag{37}$$

from which we have

$$\frac{\partial}{\partial t_m} \big(w\phi_1\big) = \frac{\partial}{\partial x} \Big(V_{31}^{(m)} \phi_1 + V_{32}^{(m)} \phi_2 + V_{33}^{(m)} \Big). \tag{38}$$

Assuming that $\rho = w\phi_1$, $\sigma = V_{31}^{(m)} \phi_1 + V_{32}^{(m)} \phi_2 + V_{33}^{(m)}$, then (38) can be written as

$$\rho_{t_m} = \sigma_x.$$

For Eq. (20), one infers

$$\rho = w\phi_1, \quad \sigma = -\big(w^2 + w_x + uw\big)\phi_1 + w\phi_2.$$

We expand ρ, σ as

$$\rho = \sum_{j=0}^{\infty} \rho_j \lambda^{-j}, \quad \sigma = \sum_{j=0}^{\infty} \sigma_j \lambda^{-j}, \tag{39}$$

where the coefficients are called conserved densities and currents respectively, then the first two conserved densities and currents read

$$\begin{aligned}
&\rho_0 = -w, \\
&\rho_1 = ww_x + wv - uw^2 - w^3, \\
&\sigma_0 = 2w^2 + w_x + uw, \\
&\sigma_1 = -w_x v - uww_x - uvw + ww_{xx} + wv_x - u_x w^2 - 4w^2 w_x - 3w^2 v \\
&\qquad + u^2 w^2 - w_x^2 + 4uw^3 + 3w^4.
\end{aligned} \tag{40}$$

The recursion relations for ρ_j and $\sigma_j (j \geq 0)$ are as follows

$$\rho_j = w\kappa_j, \quad \sigma_j = -\big(w^2 + w_x + uw\big)\kappa_j + w\chi_j, \tag{41}$$

where κ_j and χ_j can be recursively calculated from (36).

With regard to Eq. (21), we have

$$\begin{aligned} \rho &= w\phi_1, \\ \sigma &= \left(\lambda w - \tfrac{1}{2}vw + \tfrac{15}{4}ww_x + \tfrac{3}{4}u_x w + \tfrac{3}{2}uw_x + w_{xx} + \tfrac{3}{8}u^2 w + \tfrac{9}{4}uw^2 + \tfrac{15}{8}w^3\right)\phi_1 \\ &\quad - \left(\tfrac{1}{2}uw + \tfrac{3}{2}w^2 + w_x\right)\phi_2 + \lambda w. \end{aligned}$$

Expanding ρ, σ as (39), the first two conserved densities and currents read

$$\begin{aligned} \rho_0 &= -w, \\ \rho_1 &= ww_x + vw - uw^2 - w^3, \\ \sigma_0 &= \tfrac{3}{2}vw - \tfrac{15}{4}ww_x - w_{xx} - \tfrac{3}{4}u_x w - \tfrac{3}{2}uw_x - \tfrac{3}{8}u^2 w - \tfrac{15}{4}uw^2 - \tfrac{35}{8}w^3, \\ \sigma_1 &= \left(\tfrac{9}{4}uw^2 - \tfrac{1}{2}vw + \tfrac{15}{4}ww_x + \tfrac{3}{4}u_x w + \tfrac{3}{2}uw_x + \tfrac{3}{8}u^2 w + \tfrac{15}{8}w^3 + w_{xx}\right)\left(w_x + v - uw - w^2\right) \\ &\quad + w\kappa_2 - \left(\tfrac{1}{2}uw + \tfrac{3}{2}w^2 + w_x\right)\chi_1. \end{aligned} \tag{42}$$

The recursion relations for ρ_j and $\sigma_j (j \geq 0)$ are as follows

$$\begin{aligned} \rho_j &= w\kappa_j, \\ \sigma_j &= \left(\tfrac{15}{4}ww_x - \tfrac{1}{2}vw + \tfrac{3}{4}u_x w + \tfrac{3}{2}uw_x + w_{xx} + \tfrac{3}{8}u^2 w + \tfrac{9}{4}uw^2 + \tfrac{15}{8}w^3\right)\kappa_j \\ &\quad + w\kappa_{j+1} - \left(\tfrac{1}{2}uw + \tfrac{3}{2}w^2 + w_x\right)\chi_j, \end{aligned} \tag{43}$$

where κ_j and χ_j can be obtained from (36).

5 Conclusions

We proposed a new integrable system which is related to a three-component generalization of Burgers equation. By utilizing the trace identity, the bi-Hamiltonian forms of the hierarchy (19) are established. We also constructed the infinite conservation laws which are in accordance with the hierarchy (19). In the future papers, we will discuss the soliton solutions of the three-component generalization of Burgers equation by Darboux transformation, Riemann-Hilbert method, an so on. The algebro-geometric solutions of the entire hierarchy (19) may be obtained in terms of the Riemann theta function.

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Appendix

In what follows, we will prove that the operators $\widetilde{K}, \widetilde{J}$ defined in (29) are bi-Hamiltonian operators [20-24]. Since $\widetilde{K}, \widetilde{J}$ are obviously skew-symmetric, we only need to prove that $\widetilde{K}, \widetilde{J}$ fulfill the following two conditions:

(i) The Jacobi identities hold, i.e.

$$\langle \widetilde{J}'[\widetilde{J}f]g, h\rangle + \langle \widetilde{J}'[\widetilde{J}g]h, f\rangle + \langle \widetilde{J}'[\widetilde{J}h]f, g\rangle = 0, \tag{A.1a}$$

$$\langle \widetilde{K}'[\widetilde{K}f]g, h\rangle + \langle \widetilde{K}'[\widetilde{K}g]h, f\rangle + \langle \widetilde{K}'[\widetilde{K}h]f, g\rangle = 0, \tag{A.1b}$$

where $f = (f_1, f_2, f_3)^T, g = (g_1, g_2, g_3)^T$ and $h = (h_1, h_2, h_3)^T$ are vector functions.

(ii) The compatibility of two operators $\widetilde{K}, \widetilde{J}$, that is,

$$\langle \widetilde{K}'[\widetilde{J}f]g, h\rangle + \langle \widetilde{J}'[\widetilde{K}f]g, h\rangle + \text{c.p.} = 0, \tag{A.2}$$

where "c.p." denotes the cyclic permutation.

Firstly, we will prove that the operator $\widetilde{J}$ satisfies Eq. (A.1a). Denoting $\widetilde{J}f = (F_1, F_2, F_3)^T$, where

$F_1 = 6f_{1x} - 3f_{2xx} - 3(uf_2)_x - 2f_{3x}$,
$F_2 = 3f_{1xx} - 3uf_{1x} - 2f_{2xxx} - 2(uf_2)_{xx} + 2uf_{2xx} + vf_{2x} + (vf_2)_x + 2u(uf_2)_x + uf_{3x} - f_{3xx}$,
$F_3 = -2f_{1x} + (uf_2)_x + f_{2xx}$.

Then

$$\widetilde{J}'[\widetilde{J}f] = \begin{pmatrix} 0 & -3\partial F_1 & 0 \\ -3F_1\partial & -2\partial^2 F_1 + 2F_1\partial^2 + F_2\partial + \partial F_2 + 2F_1\partial u + 2u\partial F_1 & F_1\partial \\ 0 & \partial F_1 & 0 \end{pmatrix}.$$

$$\widetilde{J}'[\widetilde{J}f]g = \begin{pmatrix} -3(F_1g_2)_x \\ -3F_1g_{1x} - 4F_{1x}g_{2x} - 2F_{1xx}g_2 + 2F_2g_{2x} + F_{2x}g_2 + 4uF_1g_{2x} + 2(uF_1)_xg_2 + F_1g_{3x} \\ (F_1g_2)_x \end{pmatrix}.$$

According to the definition for the inner product:

$$\langle f, g \rangle = \int fg dx,$$

the representations of $\langle \widetilde{J}'[\widetilde{J}f]g, h\rangle, \langle \widetilde{J}'[\widetilde{J}g]h, f\rangle$ and $\langle \widetilde{J}'[\widetilde{J}h]f, g\rangle$ are generated. Furthermore, we obtain

$$\langle \widetilde{J}'[\widetilde{J}f]g, h\rangle + \langle \widetilde{J}'[\widetilde{J}g]h, f\rangle + \langle \widetilde{J}'[\widetilde{J}h]f, g\rangle = 0.$$

So the operator $\widetilde{J}$ is a Hamiltonian operator. Similarly, we can get that the Jacobi identity Eq. (A.1b) holds and the operator $\widetilde{K}$ is a Hamiltonian operator, too.

Next, we will prove that the Hamiltonian operators $\widetilde{K}$ and $\widetilde{J}$ are compatible, that is, they should satisfy Eq. (A.2). Let us define $\widetilde{K}f = (A_1, A_2, A_3)^T$ by

$$\begin{aligned} A_1 &= \widetilde{K}_{11}f_1 + \widetilde{K}_{12}f_2 + \widetilde{K}_{13}f_3, \\ A_2 &= \widetilde{K}_{21}f_1 + \widetilde{K}_{22}f_2 + \widetilde{K}_{23}f_3, \\ A_3 &= \widetilde{K}_{31}f_1 + \widetilde{K}_{32}f_2 + \widetilde{K}_{33}f_3. \end{aligned}$$

Then

$$\widetilde{J}'[\widetilde{K}f] = \begin{pmatrix} 0 & -3\partial A_1 & 0 \\ -3A_1\partial & -2\partial^2 A_1 + 2A_1\partial^2 + A_2\partial + \partial A_2 + 2A_1\partial u + 2u\partial A_1 & A_1\partial \\ 0 & \partial A_1 & 0 \end{pmatrix}.$$

$$\widetilde{J}'[\widetilde{K}f]g = \begin{pmatrix} -3(A_1g_2)_x \\ -3A_1g_{1x} - 4A_{1x}g_{2x} - 2A_{1xx}g_2 + 2A_2g_{2x} + A_{2x}g_2 + 4uA_1g_{2x} + 2(uA_1)_xg_2 + A_1g_{3x} \\ (A_1g_2)_x \end{pmatrix}.$$

$$\widetilde{K}'[\widetilde{J}f] = \left(\widetilde{K}'_{ij}\right)_{3\times 3}, \quad \widetilde{K}'[\widetilde{J}f]g = \begin{pmatrix} \widetilde{K}'_{11}g_1 + \widetilde{K}'_{12}g_2 + \widetilde{K}'_{13}g_3 \\ \widetilde{K}'_{21}g_1 + \widetilde{K}'_{22}g_2 + \widetilde{K}'_{23}g_3 \\ \widetilde{K}'_{31}g_1 + \widetilde{K}'_{32}g_2 + \widetilde{K}'_{33}g_3 \end{pmatrix},$$

where

$$\begin{aligned}
\widetilde{K}'_{11} &= -\frac{1}{2}\partial F_1\partial^{-1}u\partial - \frac{1}{2}\partial u\partial^{-1}F_1\partial,\\
\widetilde{K}'_{12} &= -3\partial^3F_1 + \partial F_1\partial^2 + \frac{3}{2}\partial^2F_2 + \frac{3}{2}\partial F_2\partial + \partial F_1\partial u + \partial u\partial F_1 - \frac{1}{2}\partial(F_1v) - \frac{1}{2}\partial(uF_2)\\
&\quad - \tfrac{1}{2}\partial u\partial^{-1}F_2\partial - \tfrac{1}{2}\partial F_1\partial^{-1}v\partial,\\
\widetilde{K}'_{13} &= 3\partial^2F_3 + \frac{3}{2}\partial F_3\partial + 2\partial F_1\partial - \partial(F_1w) - \partial(uF_3) + 2\partial F_2 - \frac{1}{2}\partial F_1\partial^{-1}w\partial - \frac{1}{2}\partial u\partial^{-1}F_3\partial,\\
\widetilde{K}'_{21} &= -3F_1\partial^3 + \partial^2F_1\partial - \frac{3}{2}F_2\partial^2 - \frac{3}{2}\partial F_2\partial - F_1\partial u\partial - u\partial F_1\partial - \frac{1}{2}uF_2\partial - \frac{1}{2}F_1v\partial\\
&\qquad - \tfrac{1}{2}\partial v\partial^{-1}F_1\partial - \tfrac{1}{2}\partial F_2\partial^{-1}u\partial,
\end{aligned}$$

$$\begin{aligned}
\widetilde{K}'_{22} &= -2\partial^4F_1 + 2F_1\partial^4 + \partial^3F_2 + F_2\partial^3 + \partial^2F_2\partial + \partial F_2\partial^2 + 2F_1\partial^3u + 2u\partial^3F_1 - u\partial^2F_2 - F_1\partial^2v\\
&\qquad + v\partial^2F_1 + F_2\partial^2u - F_1\partial v\partial - u\partial F_2\partial + \partial v\partial F_1 + \partial F_2\partial u - \partial vF_2 - vF_2\partial - \tfrac{1}{2}F_2\partial v - \tfrac{1}{2}v\partial F_2\\
&\qquad - \tfrac{1}{2}\partial F_2\partial^{-1}v\partial - \tfrac{1}{2}\partial v\partial^{-1}F_2\partial,\\
\widetilde{K}'_{23} &= 2\partial^3F_3 + F_1\partial^3 + \partial^2F_1\partial + \partial^2F_3\partial - 2F_1\partial^2w - 2u\partial^2F_3 + \partial^2F_2 - F_1\partial u\partial - u\partial F_1\partial - F_1\partial w\partial\\
&\qquad - u\partial F_3\partial - \partial(vF_3) - \partial(F_2w) - v\partial F_3 - F_2\partial w - \tfrac{1}{2}vF_3\partial - \tfrac{1}{2}F_2w\partial - u\partial F_2 - F_1\partial v\\
&\qquad - \tfrac{1}{2}\partial F_2\partial^{-1}w\partial - \tfrac{1}{2}\partial v\partial^{-1}F_3\partial,\\
\widetilde{K}'_{31} &= -3F_3\partial^2 - \frac{3}{2}\partial F_3\partial - 2\partial F_1\partial - uF_3\partial - F_1w\partial + 2F_2\partial - \frac{1}{2}\partial w\partial^{-1}F_1\partial - \frac{1}{2}\partial F_3\partial^{-1}u\partial,\\
\widetilde{K}'_{32} &= 2F_3\partial^3 + \partial^3F_1 + \partial F_1\partial^2 + \partial F_3\partial^2 + 2w\partial^2F_1 + 2F_3\partial^2u - F_2\partial^2 + \partial F_1\partial u + \partial u\partial F_1 + \partial w\partial F_1\\
&\qquad + \partial F_3\partial u - vF_3\partial - F_2w\partial - w\partial F_2 - F_3\partial v - \tfrac{1}{2}\partial(vF_3) - \tfrac{1}{2}\partial(F_2w) - F_2\partial u - v\partial F_1\\
&\qquad - \tfrac{1}{2}\partial F_3\partial^{-1}v\partial - \tfrac{1}{2}\partial w\partial^{-1}F_2\partial,\\
\widetilde{K}'_{33} &= F_3\partial^2 - \partial^2F_3 - 2\partial wF_3 - 2wF_3\partial - 2F_3\partial w - 2w\partial F_3 - uF_3\partial - F_1w\partial - \partial(uF_3) - \partial(F_1w)\\
&\qquad - \tfrac{1}{2}\partial F_3\partial^{-1}w\partial - \tfrac{1}{2}\partial w\partial^{-1}F_3\partial.
\end{aligned}$$

By some complicated calculations, we have the following result

$$\langle \widetilde{K}'[\widetilde{J}f]g, h\rangle + \langle \widetilde{J}'[\widetilde{K}f]g, h\rangle + \text{c.p.} = 0.$$

So $\widetilde{K}$ and $\widetilde{J}$ are two compatible Hamiltonian operators.

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